# Simple equations giving shapes of various convex polyhedra: the regular polyhedra and polyhedra composed of crystallographically low-index planes 

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#### Abstract

Simple equations are derived that give the shapes of various convex polyhedra. The five regular polyhedra, called Platonic solids (the tetrahedron, hexahedron or cube, octahedron, dodecahedron and icosahedron), and polyhedra composed of crystallographically low-index planes are treated. The equations also give shapes that are nearly polyhedral with round edges, or intermediate shapes between a sphere and convex polyhedra, and which are observed as the shapes of small particles or precipitates in materials.


## 1. Introduction

Small particles or precipitates in materials often show shapes similar to convex polyhedra [1, 2]. Although the origins of the shapes of materials may be explained by kinetics or energetics, in both cases the shape can be described by a simple equation that is effective in understanding the origin of the shape in a physically sound manner [2, 3]. In this letter we present such equations. Nearly polyhedral shapes with round edges or intermediate shapes between a sphere and convex polyhedra are observed for precipitates in alloys [1, 2]. The equations derived in the present study give such shapes.

The present letter is an extension of a previous two-dimensional analysis [4] in which an equation giving shapes between a circle and a regular $N$-sided polygon was presented. As typical examples of convex polyhedra, we first consider the five regular polyhedra called Platonic solids: the tetrahedron, hexahedron or cube, octahedron, dodecahedron and icosahedron. Considering crystalline materials with cubic structures, polyhedra composed of low-index planes, $\{100\},\{110\}$ and $\{111\}$, are also treated.

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## 2. The regular polyhedra

### 2.1. The hexahedron

The regular polyhedra known as Platonic solids have been described in the literature for over two thousand years. Their geometrical characteristics have been considered from many viewpoints [5-7]. However, although the Platonic solids themselves are well known, simple equations giving their shapes have not been derived in a unified form. Figure 1 shows the shapes of the five regular polyhedra and the $x-y-z$ orthogonal coordinate system to describe their shapes. We first treat the case of the hexahedron.

It is known that a solid figure described by

$$
\begin{equation*}
|x|^{p}+|y|^{p}+|z|^{p}=1 \quad(p \geq 2) \tag{1}
\end{equation*}
$$

describes a sphere when $p=2$ and a hexahedron when $p \rightarrow \infty$. This expression was presented by the 19th century French mathematician Gabriel Lamé [8]. Intermediate shapes between these two objects are called superspheres and can be represented by choosing appropriate values of $p>2$. Figure 2 shows the shapes given by equation (1) for (a) $p=2$, (b) $p=4$ and (c) $p=20$, respectively. If $|x|>|y|$ and $|x|>|z|,|x|^{p}+|y|^{p}+|z|^{p}=1$ with $p \rightarrow \infty$ means $|x|=1$. This is the reason why equation (1) with $p \rightarrow \infty$ gives a hexahedron surrounded by three sets of parallel planes, $x= \pm 1, y= \pm 1$ and $z= \pm 1$.

When $a^{2}+b^{2}+c^{2}=1, a x+b y+c z=1$ is a plane having the vector $\boldsymbol{a}=(a, b, c)$ as the unit normal vector. The orthogonal coordinate $(a, b, c)$ on this plane is the


Figure 1. The five regular polyhedra known as Platonic solids: (a) tetrahedron; (b) hexahedron; (c) octahedron; (d) dodecahedron; and (e) icosahedron. Inset: the coordinate system used to describe the shapes of the Platonic solids.
(a)

(b)

(c)


Figure 2. The shapes given by equation (1) or by equation (7) when equivalent to equation (1) when (a) $p=2$, (b) $p=4$ and (c) $p=20$. When $p \rightarrow \infty$, the shape approaches that of the hexahedron.
position of the foot of the perpendicular line from the origin. Using the function $f(a, b, c)$ given by

$$
\begin{equation*}
f(a, b, c)=a x+b y+c z \tag{2}
\end{equation*}
$$

equation (1) is rewritten as

$$
\begin{equation*}
|f(1,0,0)|^{p}+|f(0,1,0)|^{p}+|f(0,0,1)|^{p}=1 . \tag{3}
\end{equation*}
$$

In other words, when the faces of the hexahedron are written by the three sets of parallel planes as

$$
\begin{equation*}
|f(1,0,0)|=1, \quad|f(0,1,0)|=1 \quad \text { and } \quad|f(0,0,1)|=1 \tag{4}
\end{equation*}
$$

the shape of the hexahedron is written by equation (3) with $p \rightarrow \infty$.
The relationship between the orthogonal coordinates $(x, y, z)$ and the spherical coordinates $(r, \theta, \varphi)$ is

$$
\begin{equation*}
x=r \sin \theta \cos \varphi, \quad y=r \sin \theta \sin \varphi, \quad z=r \cos \theta . \tag{5}
\end{equation*}
$$

Using the spherical coordinates $(r, \theta, \varphi)$ and defining the function $g(a, b, c)$ as

$$
\begin{equation*}
g(a, b, c)=a(\sin \theta \cos \varphi)+b(\sin \theta \sin \varphi)+c(\cos \theta) \tag{6}
\end{equation*}
$$

the equation for the hexahedron is written as

$$
\begin{equation*}
r_{\text {hex }}(\theta, \varphi)=\frac{1}{\left[G_{0}(1,0,0)\right]^{1 / p}}, \tag{7}
\end{equation*}
$$

where $G_{0}(1,0,0)=|g(1,0,0)|^{p}+|g(0,1,0)|^{p}+|g(0,0,1)|^{p}$.
In the present letter, the equations giving other regular polyhedra will also be expressed using the spherical coordinates $(r, \theta, \varphi)$.

### 2.2. The octahedron, dodecahedron and icosahedron

The octahedron, dodecahedron and icosahedron are also composed of sets of parallel faces. Using the coordinate system shown in figure 1, we have the following equations for planes showing the sets of parallel faces of the regular polyhedra.

The octahedron (four sets of parallel faces):

$$
\begin{align*}
|f(\gamma, \gamma, \gamma)| & =1, \quad|f(-\gamma, \gamma, \gamma)|=1, \quad|f(\gamma,-\gamma, \gamma)|=1, \\
|f(\gamma, \gamma,-\gamma)| & =1 . \quad(\gamma=1 / \sqrt{3}) . \tag{8}
\end{align*}
$$

The dodecahedron (six sets of parallel faces):

$$
\begin{array}{r}
|f(\delta, \varepsilon, 0)|=1, \quad|f(\delta,-\varepsilon, 0)|=1, \quad|f(0, \delta, \varepsilon)|=1 \\
|f(0, \delta,-\varepsilon)|=1, \quad|f(\varepsilon, 0, \delta)|=1, \quad|f(\varepsilon, 0,-\delta)|=1  \tag{9}\\
(\delta=\sqrt{(5-\sqrt{5}) / 10}, \varepsilon=\sqrt{(5+\sqrt{5}) / 10})
\end{array}
$$

The icosahedron (ten sets of parallel faces):

$$
\begin{align*}
|f(\gamma, \gamma, \gamma)| & =1, & & |f(-\gamma, \gamma, \gamma)|=1 \\
|f(\gamma,-\gamma, \gamma)| & =1, & & |f(\gamma, \gamma,-\gamma)|=1 \tag{10}
\end{align*}
$$

and

$$
\begin{array}{r}
|f(\zeta, \eta, 0)|=1, \quad|f(\zeta,-\eta, 0)|=1, \quad|f(0, \zeta, \eta)|=1 \\
|f(0, \zeta,-\eta)|=1, \quad|f(\eta, 0, \zeta)|=1, \quad|f(\eta, 0,-\zeta)|=1  \tag{11}\\
(\zeta=\sqrt{(3-\sqrt{5}) / 6}, \eta=\sqrt{(3+\sqrt{5}) / 6})
\end{array}
$$

Equations (8) for the octahedron are identical to equations (10) for the icosahedron. Although the values are different between $(\delta, \varepsilon)$ and ( $\zeta, \eta$ ), equations (9) for the dodecahedron, it has the same form as that of equations (11) for the icosahedron.

As well as equation (7) obtained from equations (4) for the hexahedron, we have the following equations from equations (8)-(11) giving the regular polyhedra.
The octahedron:

$$
\begin{equation*}
r_{\mathrm{octa}}=\frac{1}{\left[G_{\mathrm{I}}(\gamma, \gamma, \gamma)\right]^{1 / p}}, \tag{12}
\end{equation*}
$$

where

$$
G_{\mathrm{I}}(\gamma, \gamma, \gamma)=|g(\gamma, \gamma, \gamma)|^{p}+|g(-\gamma, \gamma, \gamma)|^{p}+|g(\gamma,-\gamma, \gamma)|^{p}+|g(\gamma, \gamma,-\gamma)|^{p}
$$

The dodecahedron:

$$
\begin{equation*}
r_{\mathrm{dodeca}}=\frac{1}{\left[G_{\mathrm{II}}(\delta, \varepsilon, 0)\right]^{1 / p}}, \tag{13}
\end{equation*}
$$

where

$$
\begin{aligned}
G_{\mathrm{II}}(\delta, \varepsilon, 0)= & |g(\delta, \varepsilon, 0)|^{p}+|g(\delta,-\varepsilon, 0)|^{p}+|g(0, \delta, \varepsilon)|^{p} \\
& +|g(0, \delta,-\varepsilon)|^{\dot{p}}+|g(\varepsilon, 0, \delta)|^{p}+|g(\varepsilon, 0,-\delta)|^{p} .
\end{aligned}
$$

(a)

(b)

(c)


Figure 3. The nearly polyhedral shapes when $p=40$ for (a) the octahedron, (b) the dodecahedron and (c) the icosahedron.

The icosahedron:

$$
\begin{equation*}
r_{\mathrm{icosa}}=\frac{1}{\left[G_{\mathrm{I}}(\gamma, \gamma, \gamma)+G_{\mathrm{II}}(\zeta, \eta, 0)\right]^{1 / p}} . \tag{14}
\end{equation*}
$$

Figure 3 shows the shapes for $p=40$ given by the equations for (a) the octahedron, (b) the dodecahedron and (c) the icosahedron. The shapes given by equations (12)-(14) also change from a sphere when $p=2$ to each regular polyhedron when $p \rightarrow \infty$.

### 2.3. The tetrahedron

Different from other regular polyhedra, the tetrahedron does not have parallel faces. The function $h(a, b, c)$ given by

$$
\begin{equation*}
h(a, b, c)=\{|g(a, b, c)|-g(a, b, c)\} / 2 \tag{15}
\end{equation*}
$$

satisfies $h(a, b, c)=0$ when $g(a, b, c) \geq 0$, and $h(a, b, c)=-g(a, b, c)>0$ when $g(a, b, c)<0$. Using $h(a, b, c)$ instead of $g(a, b, c)$, we can describe the four nonparallel faces that compose the tetrahedron. That is to say,

$$
\begin{equation*}
r_{\text {tetra }-1}=\frac{1}{[H(\gamma, \gamma, \gamma)]^{1 / p}}, \tag{16}
\end{equation*}
$$

where

$$
H(\gamma, \gamma, \gamma)=\{h(\gamma, \gamma, \gamma)\}^{p}+\{h(\gamma,-\gamma,-\gamma)\}^{p}+\{h(-\gamma, \gamma,-\gamma)\}^{p}+\{h(-\gamma,-\gamma, \gamma)\}^{p}
$$

is the equation giving the tetrahedron when $p \rightarrow \infty$. However, the shape given by equation (16) is not a sphere even if $p=2$. We find that

$$
\begin{equation*}
H(\gamma, \gamma, \gamma)+H(-\gamma,-\gamma,-\gamma)=G_{\mathrm{I}}(\gamma, \gamma, \gamma) . \tag{17}
\end{equation*}
$$

Moreover, when $p \rightarrow \infty$ we find

$$
\begin{equation*}
\left[H(\gamma, \gamma, \gamma)+(1 / p)^{(p-2)} H(-\gamma,-\gamma,-\gamma)\right]^{1 / p}=[H(\gamma, \gamma, \gamma)]^{1 / p} . \tag{18}
\end{equation*}
$$



Figure 4. The shapes given by equation (19) when (a) $p=2$, (b) $p=4$ and (c) $p=14$. When $p \rightarrow \infty$, the shape approaches that of the tetrahedron.

Hence the equation

$$
\begin{equation*}
r_{\text {tetra }-2}=\frac{1}{\left[H(\gamma, \gamma, \gamma)+(1 / p)^{(p-2)} H(-\gamma,-\gamma,-\gamma)\right]^{1 / p}} \tag{19}
\end{equation*}
$$

gives a sphere when $p=2$ and the tetrahedron when $p \rightarrow \infty$. Figure 4 shows the shapes given by equation (1) when (a) $p=2$, (b) $p=4$ and (c) $p=16$, respectively.

## 3. Polyhedra composed of crystallogtaphically low-index planes

For materials with cubic structures, small particles or precipitates often show polyhedral shapes composed of crystallographically low-index planes: $\{100\},\{110\}$ and $\{111\}[9,10]$. Considering the $x, y$ and $z$ axes of the $x-y-z$ coordinate system as the $\langle 100\rangle$ axes of the cubic crystal, equations (7) and (12) with $p \rightarrow \infty$ are the equations giving the hexahedron composed of $\{100\}$ and the octahedron composed of $\{111\}$, respectively. The equation giving the rhombic dodecahedron composed of $\{110\}$ is given by equation (13) by replacing the values $(\delta, \varepsilon, 0)$ with $(\kappa, \kappa, 0)$ where $\kappa=1 / \sqrt{2}$.

The polyhedra shown in figures $5 \mathrm{a}, 5 \mathrm{~b}$ and 5 c are the hexahedron composed of $\{100\}$, the octahedron composed of $\{111\}$ and the rhombic dodecahedron composed of $\{110\}$. These, respectively, are given by the following equations with $p \rightarrow \infty$ :

$$
\begin{aligned}
r_{\text {hex }} & =\frac{1}{\left[G_{0}(1,0,0)\right]^{1 / p}}, \quad r_{\text {octa }}=\frac{1}{\left[(\sqrt{3} \alpha)^{p} G_{\mathrm{I}}(\gamma, \gamma, \gamma)\right]^{1 / p}} \\
r_{\mathrm{r}-\text { dodeca }} & =\frac{1}{\left[(\sqrt{2} \beta)^{p} G_{\mathrm{II}}(\kappa, \kappa, 0)\right]^{1 / p}}
\end{aligned}
$$

where $\alpha>0$ and $\beta>0$ are the parameters giving the size of the octahedron and the rhombic dodecahedron, respectively. Using the terms in the right-hand sides of


Figure 5. The shapes of (a) the hexahedron composed of $\{100\}$, (b) the octahedron composed of $\{111\}$, (c) the rhombic dodecahedron composed of $\{110\}$ and (d) the polyhedron composed of $\{100\}$, $\{110\}$ and $\{111\}$ given by equation (20) with $p \rightarrow \infty, \alpha=1 /(2 \sqrt{2}-1) \approx 0.55$ and $\beta=1 / \sqrt{2} \approx 0.71$.
the above equations, the equation giving the shape of a polyhedron composed of $\{100\},\{110\}$ and $\{111\}$ is written as

$$
\begin{equation*}
r=\frac{1}{\left[G_{0}(1,0,0)+(\sqrt{3} \alpha)^{p} G_{\mathrm{I}}(\gamma, \gamma, \gamma)+(\sqrt{2} \beta)^{p} G_{\mathrm{II}}(\kappa, \kappa, 0)\right]^{1 / p}} . \tag{20}
\end{equation*}
$$

The parameters $\alpha$ and $\beta$ in the right-hand side of equation (20) change the shape of the polyhedron composed of $\{100\},\{110\}$ and $\{111\}$. The shape given by equation (20) for $\alpha=1 /(2 \sqrt{2}-1) \approx 0.55$ and $\beta=1 / \sqrt{2} \approx 0.71$ is shown in figure 5 d . This polyhedron has six square $\{100\}$, twelve square $\{110\}$ and eight equilateral-triangular $\{111\}$. To show the relationship between the polyhedra shown in figures 5 a to 5 d , the common points, P and Q on the polyhedra are also indicated.

Figure 6 is a map showing the variation of the shape of polyhedron given by equation (20) with $p \rightarrow \infty$. The shapes of the polyhedra in various regions are shown by insets. The region for the hexahedron with $\{100\}$ is that of $\alpha \leq 1 / 3$ and $\beta \leq 1 / 2$, the one for the rhombic dodecahedron with $\{110\}$ is that of $1 \leq \beta$ and $3 \alpha / 2 \leq \beta$, and


Figure 6. Map showing the variations of the shape of the $\{100\}-\{110\}-\{111\}$ polyhedron given by equation (20) with $p \rightarrow \infty$.
the one for the octahedron with $\{111\}$ is that of $1 \leq \alpha$ and $\beta \leq \alpha$. Other regions are for the polyhedra composed of two or three kinds of the low-index planes. For example, when $\beta=1 / 2$, the shape changes from the hexahedron to the octahedron with the increase of $\alpha$ from $1 / 3$ to 1 .

Coherent precipitates with misfit strains cause elastic deformation of material containing the precipitates. The elastic strain energy caused by misfit strains depends on the precipitate shape $[2,3,11]$. To calculate the shape dependence of the elastic strain energy, an equation describing various shapes of polyhedral precipitates such as equation (18) is a useful tool $[2,3]$. Moreover, using certain values of $p$, we can easily express the effects of round edges of the polyhedral precipitates [2, 3]. Application of the present results to such energy considerations will be the subject of our future work.

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